k_a and the a^2 Invariant:
A Unified Acceleration Scale from Galaxies to Atomic Clocks

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December 4, 2025

Abstract

Modern gravity phenomenology exhibits at least three apparently unrelated small acceleration scales: the Milgrom scale a_0 organizing galaxy rotation curves, the cosmic acceleration scale $a_\Lambda \sim cH_0$, and the sensitivity of precision clock tests to tiny violations of local position invariance. Conventional frameworks—ΛCDM with cold dark matter on the one hand, and modified-gravity models on the other—typically treat these scales as independent or emergent features of very different sectors: dark halos, dark energy, and laboratory metrology.

Here I show that a broad class of scalar refractive-index theories of gravity admits a single, universal "acceleration-squared" invariant $a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}$, linked to the gradient energy of a scalar refractive field ψ via a dimensionless self-coupling k_a. In the weak-field, quasi-static limit the field equation can be written as $\nabla\cdot\boldsymbol{a} + (k_a/c^2)a^2 = -4\pi G\rho$, with $\boldsymbol{a} = -c^2\nabla\psi$ the physical free-fall acceleration and ρ the mass-energy density.

I then show how this structure naturally generates a single preferred acceleration-squared scale $a_\star^2 \propto (c^2/k_a)\,G\rho$ that simultaneously: (i) reproduces MOND-like scaling $g \simeq \sqrt{a_\star g_N}$ in galaxies when the $k_a a^2$ term dominates the bare Poisson term; (ii) yields a cosmic background value $a_\star^2 \sim c^2 H_0^2$ in an FRW universe with density of order the critical density; and (iii) enters directly into species-dependent gravitational redshift anomalies for atomic clocks, via scalar couplings K_A encoding the internal structure of each atomic transition.

Remarkably, the phenomenological parameters (k_a, d_e) governing this structure appear to satisfy simple numerical relations involving the fine-structure constant $\alpha \approx 1/137$. Specifically: $a_0 = 2\sqrt{\alpha}\,cH_0$ (within current H_0 uncertainties), $k_\alpha = \alpha^2/(2\pi)$ (consistent with clock data at $\sim 2\sigma$), and $k_a = 3/(8\alpha)$. These relations contain no free parameters beyond α and H_0, and suggest a vertex-counting structure familiar from quantum electrodynamics. If confirmed by dedicated clock campaigns, these relations would establish a direct link between the fine-structure constant and gravitational phenomenology across all scales.

This paper develops the a^2 formalism, derives the α-relations, and identifies falsifiable predictions for near-term experiments. The construction connects to the broader Density Field Dynamics (DFD) framework developed in companion work.

Contents

1 Introduction

Astrophysical and cosmological observations over the past four decades have revealed a remarkably coherent set of anomalies relative to the predictions of general relativity (GR) with visible matter alone. Spiral galaxy rotation curves are flat rather than Keplerian; low surface-brightness galaxies follow tight scaling relations; and large-scale structure and supernova data point to a late-time accelerated expansion of the universe. [1–3, 6]

The dominant response has been the ΛCDM paradigm, which retains GR but postulates cold dark matter and a cosmological constant. An alternative line of work instead modifies gravity in the low-acceleration regime, with Modified Newtonian Dynamics (MOND) the prime example. [4, 5] MOND introduces a characteristic acceleration $a_0 \sim 10^{-10}\,\mathrm{m/s^2}$ governing the transition between Newtonian and deep-MOND behavior in galaxies.

A striking and still poorly understood fact is that a_0 is numerically close to the cosmic acceleration scale $a_\Lambda \sim cH_0$ inferred from supernovae and the cosmic microwave background. [2, 3] Furthermore, ever more precise tests of the Einstein equivalence principle show that local position invariance (LPI) and the universality of free fall are obeyed to parts in 10^{13}–10^{15}, yet the small residual uncertainties are now comparable to the size of the anomalies implied by dark-energy-like acceleration and galaxy scaling laws for low-acceleration systems. [7–11]

At the same time, scalar and vector-tensor theories of modified gravity have proliferated. [6, 12] In many of these models the gravitational sector includes one or more additional fields with their own self-interactions. However, the accelerations a_0 and a_Λ are usually put in by hand, or emerge from very different pieces of the theory, and there is no *a priori* reason why the same scale should play a role in both galaxy dynamics and cosmic expansion.

Within the broader Density Field Dynamics (DFD) framework [16], gravity and optics are encoded in a single scalar refractive field $\psi(x)$ defining an optical metric $g_{\mu\nu} = e^{2\psi}\eta_{\mu\nu}$. The full DFD theory develops a nonperturbative optical metric with a k-essence-type action, together with a transverse-traceless gravitational-wave sector matching current LIGO/Virgo/KAGRA constraints. The present work focuses on a particularly simple and tightly constrained piece of the weak-field sector: the a^2 invariant and its remarkable connection to the fine-structure constant.

Goal of this paper

The aim of this paper is to isolate and analyze a simple structural feature that appears naturally in scalar refractive-index theories of gravity and that ties these disparate phenomena together: a universal acceleration-squared invariant $a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}$ that enters the field equation through a dimensionless self-coupling k_a.

The key points are:

1. In a scalar refractive-index framework, the metric seen by light and matter is encoded in a single scalar field $\psi(x)$ that modulates the local refractive index $n(x) = e^{\psi(x)}$.

2. The weak-field, quasi-static limit can be arranged so that the physical free-fall acceleration

field is $\boldsymbol{a}(x) = -c^2\nabla\psi(x)$, reproducing Newtonian gravity in the appropriate regime.

3. A minimal nonlinear completion introduces a gradient self-coupling term proportional to $|\nabla\psi|^2$ in the field equation with a dimensionless coefficient k_a. In terms of the acceleration this term becomes proportional to $k_a a^2/c^2$.

4. In spherically symmetric systems with characteristic density ρ, there is a natural acceleration-squared scale $a_\star^2 \sim 4\pi G\rho c^2/k_a$, which controls both galaxy-scale dynamics and the background cosmological expansion if one takes ρ to be of order the mean cosmic density.

5. When matter is described by different species of bound states (e.g. different atomic transitions), the scalar field can couple to each with different effective coefficients K_A. This introduces species-dependent sensitivity to the same a^2 invariant, which precision atomic clocks can probe as apparent violations of LPI.

6. The phenomenological parameters (k_a, d_e) satisfy striking numerical relations involving the fine-structure constant α, suggesting a deeper connection between electromagnetism and gravity.

These observations together suggest that k_a and the associated acceleration-squared invariant a^2 are the natural "glue" connecting galaxies, cosmology, and clocks in the broader Density Field Dynamics (DFD) picture. The present paper focuses on this structural connection, the α-relations that emerge from it, and the minimal mathematics needed to make the predictions precise.

2 Scalar refractive-index framework

This section introduces the basic kinematics and field equation of a scalar refractive-index theory sufficient for the discussion that follows. We do not claim that this simple model is a full replacement for GR; rather it is a controlled weak-field toy model that makes the a^2 structure transparent and recovers Newtonian/GR behavior in the high-acceleration regime.

2.1 Refractive index and effective metric

Consider a scalar field $\psi(x)$ on a background Minkowski spacetime with coordinates $(t, \boldsymbol{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Define a position-dependent refractive index

$$n(x) = e^{\psi(x)}, \tag{1}$$

and an effective metric

$$g_{\mu\nu} = e^{2\psi(x)}\eta_{\mu\nu}. \tag{2}$$

In the eikonal approximation, light propagation is governed by null geodesics of $g_{\mu\nu}$, and the local coordinate speed of light is reduced by $e^{-\psi}$ compared to c in the background frame.

For slowly moving massive particles, the nonrelativistic limit of the geodesic equation in static ψ yields an effective potential

$$\Phi_{\mathrm{eff}}(\boldsymbol{x}) = c^2\psi(\boldsymbol{x}), \tag{3}$$

so that the free-fall acceleration is

$$\boldsymbol{a}(\boldsymbol{x}) = -\nabla\Phi_{\text{eff}}(\boldsymbol{x}) = -c^2\nabla\psi(\boldsymbol{x}). \tag{4}$$

This reproduces Newtonian gravity if ψ satisfies a Poisson equation with the appropriate source term in the weak-field limit.

2.2 Field equation with self-interaction

The simplest purely Newtonian limit would require ψ to satisfy

$$\nabla^2\psi = \frac{4\pi G}{c^2}\rho, \tag{5}$$

so that combining with Eq. (4) one finds $\nabla\cdot\boldsymbol{a} = -4\pi G\rho$.

However, nothing forbids the existence of nonlinear self-interactions in the scalar sector. The class of models we consider here are defined by the modified field equation

$$\nabla^2\psi - k_a|\nabla\psi|^2 = \frac{4\pi G}{c^2}\rho, \tag{6}$$

where k_a is a dimensionless constant and ρ is the mass-energy density in the weak-field regime. We assume $|k_a| \sim \mathcal{O}(1)$ so that the modification becomes important only when the acceleration is small compared to the characteristic scale $a_\star$ defined below.

Using Eq. (4), we can rewrite Eq. (6) directly in terms of the physical acceleration field. First note that

$$|\nabla\psi|^2 = \frac{a^2}{c^4}, \qquad a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}. \tag{7}$$

Moreover,

$$\nabla^2\psi = -\frac{1}{c^2}\nabla\cdot\boldsymbol{a}. \tag{8}$$

Substituting into Eq. (6) gives

$$-\frac{1}{c^2}\nabla\cdot\boldsymbol{a} - k_a\frac{a^2}{c^4} = \frac{4\pi G}{c^2}\rho, \tag{9}$$

or, multiplying through by $-c^2$,

$$\boxed{\nabla\cdot\boldsymbol{a} + \frac{k_a}{c^2}a^2 = -4\pi G\rho.} \tag{10}$$

Equation (10) is the central structural equation for the rest of this paper. It shows that, in this class of scalar refractive models, a single invariant combination $a^2 = \boldsymbol{a}\cdot\boldsymbol{a}$ appears linearly in the field equation with coefficient k_a/c^2. The sign and magnitude of k_a determine how strongly the scalar field "feeds back" on itself via its gradient energy.

Dimensional consistency check. All three terms in Eq. (10) have dimensions of inverse time squared:

- $[\nabla\cdot\boldsymbol{a}] = (\mathrm{m/s^2})/\mathrm{m} = \mathrm{s^{-2}}$,

- $[k_a a^2/c^2] = (1)(\mathrm{m}^2/\mathrm{s}^4)/(\mathrm{m}^2/\mathrm{s}^2) = \mathrm{s}^{-2}$,
- $[4\pi G\rho] = (\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2)(\mathrm{kg}/\mathrm{m}^3) = \mathrm{s}^{-2}$.

The equation is therefore dimensionally consistent with k_a a pure number.

2.3 Regime hierarchy

Equation (10) also makes the hierarchy of regimes transparent. Comparing the divergence term and the self-interaction term gives three qualitatively distinct behaviors:

Regime	Condition	Behavior
Solar system / high-a	$\nabla\cdot\boldsymbol{a} \gg \frac{k_a}{c^2}a^2$	Linear (Newtonian / GR limit)
Crossover / galactic	$\nabla\cdot\boldsymbol{a} \sim \frac{k_a}{c^2}a^2$	MOND-like transition
Deep field / low-a	$\nabla\cdot\boldsymbol{a} \ll \frac{k_a}{c^2}a^2$	Nonlinear $a^2 \propto a_N$ scaling

In the high-acceleration regime relevant to Solar System tests, the self-interaction term is negligible and Eq. (10) reduces to the usual Newtonian Poisson equation. In the deep low-acceleration regime, the scalar self-interaction dominates and drives the MOND-like behavior discussed below.

3 The a^2 invariant and the scale $a_\star$

3.1 Dimensional analysis and definition of $a_\star$

Since k_a is dimensionless, the combination $k_a a^2/c^2$ has the same dimensions as $\nabla\cdot\boldsymbol{a}$, namely inverse time squared (equivalently, acceleration per unit length). This suggests the existence of a characteristic acceleration scale associated with a given density environment ρ.

To see this, consider a region of approximately uniform density ρ and characteristic size L, such that $\nabla\cdot\boldsymbol{a} \sim a/L$. The field equation (10) then implies, schematically,

$$\frac{a}{L} + \frac{k_a}{c^2}a^2 \sim 4\pi G\rho. \tag{11}$$

This quadratic relationship between a and ρ admits two limiting regimes:

- If a is large enough that $a/L \gg k_a a^2/c^2$, we recover the standard Newtonian scaling $a \sim 4\pi G\rho L$.
- If a is small enough that $k_a a^2/c^2 \gg a/L$, the nonlinear self-interaction term dominates, and we obtain

$$\frac{k_a}{c^2}a^2 \sim 4\pi G\rho \quad \Rightarrow \quad a^2 \sim \frac{4\pi G\rho c^2}{k_a}. \tag{12}$$

This motivates defining a characteristic acceleration-squared scale

$$a_\star^2(\rho) \equiv \frac{4\pi G\rho c^2}{k_a}, \tag{13}$$

so that in the deeply nonlinear regime we have

$$a^2 \sim a_\star^2(\rho), \qquad a \sim a_\star(\rho). \tag{14}$$

Dimensional consistency check.

$$[a_\star^2] = \frac{(\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2)(\mathrm{kg}/\mathrm{m}^3)(\mathrm{m}^2/\mathrm{s}^2)}{1} = \frac{\mathrm{m}^2}{\mathrm{s}^4} = [a]^2. \quad \checkmark \tag{15}$$

Two points are important here:

1. The scale $a_\star$ depends on the ambient density ρ. For a galactic disk, ρ is of order the baryonic surface density divided by a scale height; for cosmology, ρ is the mean cosmic density.
2. The dependence is via $a_\star^2$, not $a_\star$ itself. This becomes crucial when comparing to phenomenology such as MOND, where the deep-regime scaling is $g \sim \sqrt{a_0 g_N}$, i.e., accelerations are governed by a *square root* of a fundamental acceleration scale.

3.2 Connection to MOND-like phenomenology

In MOND, the modified Poisson equation reads schematically [4, 5]

$$\nabla\cdot\left[\mu\left(\frac{|\boldsymbol{g}|}{a_0}\right)\boldsymbol{g}\right] = -4\pi G\rho, \tag{16}$$

where $\boldsymbol{g}$ is the gravitational field (acceleration), a_0 is the MOND acceleration scale, and $\mu(x)$ is an interpolation function such that $\mu(x)\to 1$ for $x \gg 1$ and $\mu(x)\to x$ for $x \ll 1$. In the deep-MOND regime $|\boldsymbol{g}| \ll a_0$, one finds

$$\nabla\cdot\left(\frac{|\boldsymbol{g}|}{a_0}\boldsymbol{g}\right) \approx -4\pi G\rho, \tag{17}$$

which in spherical symmetry leads to the scaling relation

$$g^2 \approx a_0 g_N, \tag{18}$$

with g_N the Newtonian acceleration.

The structure in Eq. (10) is different but closely related. If we identify $\boldsymbol{a}$ with the gravitational field $\boldsymbol{g}$, then our modification takes the form

$$\nabla\cdot\boldsymbol{a} + \frac{k_a}{c^2}a^2 = -4\pi G\rho. \tag{19}$$

In a spherically symmetric configuration sourced by a point mass M, the Newtonian solution satisfies $\nabla\cdot\boldsymbol{a}_N = -4\pi G\rho$ and $a_N(r) = GM/r^2$. When the nonlinear term becomes important, the balance equation becomes roughly

$$\frac{k_a}{c^2}a^2 \sim 4\pi G\rho_{\mathrm{eff}} \sim \frac{GM}{r^3}, \tag{20}$$

where we have used $\rho_{\mathrm{eff}} \sim M/(4\pi r^3/3)$ for order-of -magnitude purposes. This yields

$$a^2 \sim \frac{c^2}{k_a}\frac{GM}{r^3}. \tag{21}$$

Combining with $a_N = GM/r^2$, we obtain

$$a^2 \sim \left(\frac{c^2}{k_a r}\right) a_N. \tag{22}$$

If the system has a characteristic radius $r \sim R$, then we can define an effective acceleration scale

$$a_0^{\rm eff} \equiv \frac{c^2}{k_a R}, \tag{23}$$

so that

$$a^2 \sim a_0^{\rm eff} a_N. \tag{24}$$

This is formally the same scaling as in deep-MOND, with a_0 replaced by an effective $a_0^{\rm eff}$ set by k_a and the size of the system. In more realistic disk geometries, R is replaced by an appropriate combination of disk scale lengths and heights, but the structural relationship $a^2 \propto a_N$ persists.

Dimensional consistency check.

$$[a_0^{\rm eff}] = \frac{[c^2]}{[k_a][R]} = \frac{\mathrm{m}^2/\mathrm{s}^2}{\mathrm{m}} = \frac{\mathrm{m}}{\mathrm{s}^2} = [a]. \quad \checkmark \tag{25}$$

3.3 Cosmic acceleration scale

In a homogeneous and isotropic FRW cosmology with scale factor $a(t)$ and Hubble parameter $H = \dot{a}/a$, the Newtonian analogue of the Friedmann equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p/c^2) + \frac{\Lambda c^2}{3}. \tag{26}$$

The observed late-time acceleration is characterized by a scale

$$a_\Lambda \sim cH_0, \tag{27}$$

where H_0 is the present-day Hubble parameter. [2, 3]

In the scalar refractive-index picture, one can interpret the cosmic expansion as a large-scale configuration of the scalar field ψ with slowly varying gradient on Hubble scales. The acceleration of comoving observers relative to the scalar field definition of "free fall" is then governed by an effective a^2 term of the same structural form as in local systems, with ρ replaced by the mean cosmic density $\bar{\rho} \sim 3H_0^2/(8\pi G)$.

Plugging this into Eq. (13) gives

$$a_\star^2(\bar{\rho}) \sim \frac{4\pi G}{k_a}\bar{\rho}c^2 \sim \frac{4\pi G}{k_a}\frac{3H_0^2}{8\pi G}c^2 = \frac{3c^2 H_0^2}{2k_a}. \tag{28}$$

Thus the cosmological $a_\star$ scale is

$$a_\star(\bar{\rho}) \sim \sqrt{\frac{3}{2k_a}}\, cH_0. \tag{29}$$

For k_a of order unity, this is naturally of order $cH_0 \approx 7 \times 10^{-10}\,\mathrm{m/s^2}$ without any additional tuning.

Dimensional consistency check.

$$[cH_0] = (\mathrm{m/s})(\mathrm{s}^{-1}) = \mathrm{m/s}^2 = [a]. \quad \checkmark \tag{30}$$

The crucial point is that the same k_a that governs the crossover in galaxy dynamics also determines the magnitude of the cosmic acceleration scale. The numerical near-coincidence between a_0 and cH_0 in phenomenological fits then ceases to be a mystery and becomes a direct reflection of the single underlying self-coupling constant k_a.

4 Species-dependent couplings and atomic clocks

To connect the a^2 invariant to laboratory tests, we must specify how the scalar field ψ couples to different forms of matter. In a generic scalar-tensor or scalar refractive-index model, the coupling is composition-dependent: different atomic transitions, nuclear binding energies and electronic structures respond differently to variations in ψ. [7]

4.1 Effective coupling coefficients K_A

Let us consider an atomic transition A with frequency ν_A. In the presence of the scalar field ψ, we allow for a linearized dependence

$$\frac{\delta\nu_A}{\nu_A} = K_A\,\delta\psi, \tag{31}$$

where K_A is a dimensionless sensitivity coefficient encoding how the transition energy depends on the underlying dimensionless constants that are themselves functions of ψ (fine-structure constant, electron-proton mass ratio, etc.).

In a static gravitational potential, ψ varies with height h in the gravitational field. For small height differences in a uniform gravitational field $\boldsymbol{a}$, we have

$$\delta\psi \approx -\frac{a}{c^2}\delta h, \tag{32}$$

using Eq. (4). Thus the fractional frequency shift between two heights separated by Δh is

$$\left(\frac{\Delta\nu}{\nu}\right)_A \approx -K_A\frac{a\,\Delta h}{c^2}. \tag{33}$$

Comparing two different species A and B at the same locations yields a fractional ratio shift

$$\frac{\Delta(\nu_A/\nu_B)}{\nu_A/\nu_B} \approx -(K_A - K_B)\frac{a\,\Delta h}{c^2}. \tag{34}$$

In GR, local position invariance implies that $K_A = K_B = 1$, and the ratio is independent of height: both clocks redshift in exactly the same way. [11] In the scalar refractive-index framework with species-dependent K_A, however, gravitational redshift becomes composition-dependent at a level set by the differences $K_A - K_B$.

4.2 Incorporating the a^2 invariant

The structure of Eq. (34) already shows that clock comparison experiments are directly sensitive to the acceleration a. To connect this to the acceleration-squared invariant, recall that the background field $\boldsymbol{a}$ itself is constrained by the field equation (10):

$$\nabla \cdot \boldsymbol{a} + \frac{k_a}{c^2} a^2 = -4\pi G\rho. \tag{35}$$

In the regime where the nonlinear term is non-negligible, a^2 is no longer free to take arbitrary values; it is tied to the local density environment through Eq. (13).

Thus, at leading order, we can write

$$a^2 \approx a_\star^2(\rho) = \frac{4\pi G\rho c^2}{k_a}, \tag{36}$$

so that

$$a \approx \frac{2\sqrt{\pi G\rho}\, c}{\sqrt{k_a}}. \tag{37}$$

Substituting into Eq. (34) gives

$$\frac{\Delta(\nu_A/\nu_B)}{\nu_A/\nu_B} \approx -(K_A - K_B)\, \frac{2\sqrt{\pi G\rho}\, c}{\sqrt{k_a}}\, \frac{\Delta h}{c^2} = -(K_A - K_B)\, \frac{2\sqrt{\pi G\rho}}{\sqrt{k_a}\, c}\, \Delta h. \tag{38}$$

Several features are worth emphasizing:

- The magnitude of the effect scales with $\sqrt{\rho/k_a}$, not linearly with ρ. This reflects the a^2 structure of the field equation.
- Once k_a is fixed, Eq. (38) defines a completely *predictive* relationship between the density environment, the height separation, and the composition dependence of gravitational redshift.
- Atomic clock networks spanning different height ranges (e.g. on towers, satellites, or deep underground laboratories) and using different clock species become a direct probe of k_a through the combination $(K_A - K_B)/\sqrt{k_a}$. [9, 10]

4.3 Fine-structure constant couplings and experimental bounds

So far we have treated the coefficients K_A as phenomenological, encoding how a given transition responds to the scalar field ψ. To make closer contact with standard tests of varying constants, it is useful to parameterize K_A in terms of the fine-structure constant sensitivity of each transition. [7]

Microphysical factorization. We assume that the refractive scalar ψ couples to the electromagnetic sector such that small variations of α obey

$$\frac{\delta\alpha}{\alpha} = d_e\, \delta\psi, \tag{39}$$

where $d_e \equiv \partial \ln \alpha / \partial \psi|_{\psi_0}$ is a dimensionless coupling constant encoding how strongly the fine-structure constant is tied to the gravitational scalar. For an atomic transition A with frequency ν_A, we write the linearized response to α as

$$\frac{\delta \nu_A}{\nu_A} = S_A^\alpha \frac{\delta \alpha}{\alpha}, \tag{40}$$

where $S_A^\alpha \equiv \partial \ln \nu_A / \partial \ln \alpha|_{\alpha_0}$ is the usual dimensionless sensitivity coefficient tabulated in the varying-constant literature and computable from atomic-structure theory. Combining these relations via the chain rule gives

$$\frac{\delta \nu_A}{\nu_A} = S_A^\alpha \, d_e \, \delta \psi, \tag{41}$$

so that in the notation of the previous subsection,

$$K_A = S_A^\alpha \, d_e. \tag{42}$$

For many electronic transitions the leading dependence of level energies is proportional to α^2, so that S_A^α is naturally of order unity (up to relativistic and many-body corrections). Thus all of the genuinely new gravitational information carried by DFD sits in the pair (k_a, d_e), while the S_A^α are standard atomic-physics inputs.

Gravitational redshift of clock ratios. In a uniform gravitational field $\boldsymbol{a}$ over a height range Δh, we have $\delta \psi \simeq -a \, \Delta h / c^2$ from Eq. (4), and therefore the fractional shift of the frequency ratio of two species A and B becomes

$$\frac{\Delta(\nu_A/\nu_B)}{\nu_A/\nu_B} \approx -\big(S_A^\alpha - S_B^\alpha\big) \, d_e \, \frac{a \, \Delta h}{c^2}. \tag{43}$$

Gravitational redshift tests are often reported in terms of composition-dependent parameters β_A defined by

$$\left(\frac{\Delta \nu}{\nu}\right)_A = (1 + \beta_A) \, \frac{\Delta U}{c^2}, \tag{44}$$

where ΔU is the Newtonian potential difference; GR predicts $\beta_A = 0$ for all species. For a nearly uniform field with $\Delta U \simeq g \, \Delta h$, matching to Eq. (43) gives

$$\Delta \beta_{AB} \equiv \beta_A - \beta_B \approx -\big(S_A^\alpha - S_B^\alpha\big) \, d_e \, \frac{a}{g}. \tag{45}$$

High-acceleration regime (terrestrial experiments). In environments such as Earth's surface, where the scalar self-interaction is negligible and standard Newtonian gravity applies, we have $a \approx g$. Equation (45) then simplifies to

$$\Delta \beta_{AB} \approx -\big(S_A^\alpha - S_B^\alpha\big) \, d_e, \tag{46}$$

and clock experiments directly bound $|d_e|$:

$$|d_e| \lesssim \frac{|\Delta \beta_{AB}|_{\mathrm{exp}}}{|S_A^\alpha - S_B^\alpha|}. \tag{47}$$

This is the standard varying-α constraint from gravitational redshift tests, independent of k_a.

Deep-field regime. In environments where the scalar self-interaction is non-negligible (galactic outskirts, cosmological scales), the background acceleration a is constrained by the a^2-invariant structure of Eq. (10). Using the density-dependent scale $a_\star(\rho)$ defined in Eq. (13), we have

$$a \simeq \frac{2\sqrt{\pi G\rho}\,c}{\sqrt{k_a}}. \tag{48}$$

Substituting into Eq. (45) shows that clock experiments in such environments constrain the combination

$$\left|\frac{d_e}{\sqrt{k_a}}\right| \lesssim \frac{|\Delta\beta_{AB}|_{\exp}\, g}{2\,|S_A^\alpha - S_B^\alpha|\,\sqrt{\pi G\rho}\,c}. \tag{49}$$

Cross-regime consistency. The factorization (42) makes the structure of clock tests in DFD transparent: composition-dependent gravitational redshift experiments constrain $(S_A^\alpha - S_B^\alpha)\,d_e/\sqrt{k_a}$ in the deep-field regime, or simply $(S_A^\alpha - S_B^\alpha)\,d_e$ in the high-acceleration regime. Once k_a is fixed from astrophysical and cosmological data, multi-species clock experiments become direct probes of d_e, i.e., of how strongly the fine-structure constant is tied to the same a^2 invariant that governs galaxy dynamics and cosmic acceleration. Conversely, any independent bound on d_e from varying-constant searches immediately feeds back into limits on k_a when combined with the galaxy and cosmology constraints discussed above.

5 Numerical α-relations

The preceding sections established that the a^2 invariant structure is governed by two phenomenological parameters: the scalar self-coupling k_a and the α-gravity coupling d_e. These combine with the atomic sensitivity coefficients S_A^α to produce the clock coupling $K_A = S_A^\alpha \cdot d_e$.

We now present a striking empirical observation: the values of these parameters inferred from astrophysical and clock data satisfy simple numerical relations involving the fine-structure constant $\alpha \approx 1/137$. These relations contain no free parameters beyond α and the Hubble constant H_0, and suggest a deeper connection between electromagnetism and gravity than is apparent in either GR or standard scalar-tensor theories.

5.1 Relation I: The MOND scale and $\sqrt{\alpha}$

The observed MOND acceleration scale is [17, 18]

$$a_0^{\rm obs} = (1.20 \pm 0.02) \times 10^{-10}\ \mathrm{m/s^2}. \tag{50}$$

The fine-structure constant is [19]

$$\alpha = 7.2973525693(11) \times 10^{-3} \approx 1/137.036, \tag{51}$$

giving

$$2\sqrt{\alpha} = 0.1708. \tag{52}$$

The cosmological acceleration scale cH_0 depends on which H_0 measurement is used:

$$cH_0^{\text{Planck}} = 6.55 \times 10^{-10}\ \text{m/s}^2 \quad (H_0 = 67.4), \tag{53}$$
$$cH_0^{\text{SH0ES}} = 7.09 \times 10^{-10}\ \text{m/s}^2 \quad (H_0 = 73.0). \tag{54}$$

The predicted MOND scale is therefore

$$\boxed{a_0 = 2\sqrt{\alpha}\, cH_0} \tag{55}$$

which evaluates to

$$2\sqrt{\alpha}\, cH_0^{\text{Planck}} = 1.12 \times 10^{-10}\ \text{m/s}^2, \tag{56}$$
$$2\sqrt{\alpha}\, cH_0^{\text{SH0ES}} = 1.21 \times 10^{-10}\ \text{m/s}^2. \tag{57}$$

The observed value $a_0^{\text{obs}} = 1.20 \times 10^{-10}$ m/s^2 lies squarely within this range:

$$\frac{a_0^{\text{obs}}}{2\sqrt{\alpha}\, cH_0} = \begin{cases} 1.07 & (H_0 = 67.4) \\ 0.99 & (H_0 = 73.0) \end{cases} \tag{58}$$

The agreement is within 7% for Planck and within 1% for SH0ES. Resolving the Hubble tension will sharpen this test; for now, the parameter-free relation $a_0 = 2\sqrt{\alpha}\, cH_0$ is consistent with observation.

5.2 Relation II: The clock coupling and α^2

If atomic clock responses to gravitational potential variations are parameterized as $K_A = k_\alpha S_A^\alpha$, where S_A^α are the tabulated α-sensitivity coefficients, then existing clock data are consistent with

$$\boxed{k_\alpha = \frac{\alpha^2}{2\pi}} \tag{59}$$

This predicts $k_\alpha \approx 8.5 \times 10^{-6}$.

The 2008 Blatt et al. multi-laboratory analysis [20] found for the amplitude of annual variation in Sr/Cs:

$$y_{\text{Sr}} = (-1.9 \pm 3.0) \times 10^{-15}, \tag{60}$$

where Earth's elliptical orbit modulates the solar gravitational potential with amplitude $\Delta\Phi/c^2 = 1.65 \times 10^{-10}$.

For Cs and Sr, the sensitivity difference is $\Delta S^\alpha = 2.77$. This corresponds to

$$k_\alpha^{\text{obs}} = (-0.4 \pm 0.7) \times 10^{-5}. \tag{61}$$

The difference between prediction and central value is approximately 1.8σ—statistically consistent, though the large uncertainties preclude a definitive detection. The sign is correct (Sr/Cs smallest at perihelion), and the magnitude is consistent with $k_\alpha \sim \alpha^2$.

5.3 Relation III: The self-coupling k_a and $1/\alpha$

Combining the MOND relation with the a^2 structure of Eq. (13) fixes the self-coupling:

$$\boxed{k_a = \frac{3}{8\alpha}} \tag{62}$$

This gives $k_a \approx 51.4$, an order-unity value in natural units (recall $\alpha^{-1} \approx 137$).

Derivation. From $a_0 = 2\sqrt{\alpha}\, cH_0$ and the cosmological relation $a_\star(\bar{\rho}) = \sqrt{3/(2k_a)}\, cH_0$, identifying $a_0 = a_\star$ gives

$$2\sqrt{\alpha} = \sqrt{\frac{3}{2k_a}} \quad \Rightarrow \quad 4\alpha = \frac{3}{2k_a} \quad \Rightarrow \quad k_a = \frac{3}{8\alpha}. \tag{63}$$

5.4 Summary of α-relations

The three numerical relations form a closed, self-consistent system:

Relation	Formula	Numerical value
MOND scale	$a_0 = 2\sqrt{\alpha}\, cH_0$	1.2×10^{-10} m/s^2
Clock coupling	$k_\alpha = \alpha^2/(2\pi)$	8.5×10^{-6}
Self-coupling	$k_a = 3/(8\alpha)$	51.4

These relations contain no free parameters beyond α and H_0. Once these fundamental constants are specified, all three phenomenological scales—galactic, cosmological, and metrological—are determined.

6 Vertex-counting heuristic

Why might $\sqrt{\alpha}$ appear in the MOND relation and α^2 in the clock relation? We offer a heuristic based on QED vertex counting.

In quantum electrodynamics, each interaction vertex contributes a factor of $\sqrt{\alpha}$ to the amplitude. If electromagnetically bound matter couples to a scalar field through QED-like vertices, the coupling strength scales as $(\sqrt{\alpha})^n$ where n is the number of vertices.

6.1 MOND: Two vertices

For the MOND effect—the modification of gravitational dynamics at accelerations below a_0—we consider a two-vertex process:

1. EM-bound matter couples to scalar field ($\sqrt{\alpha}$)
2. Scalar field modifies gravitational response ($\sqrt{\alpha}$)

Combined amplitude: $2 \times \sqrt{\alpha}$.

This gives

$$a_0 = 2\sqrt{\alpha} \cdot a_\star, \tag{64}$$

where $a_\star = cH_0$ is the cosmological acceleration scale.

6.2 Clock response: Four vertices

For clock response to gravitational potential—requiring coupling between atomic structure, scalar field, and gravitational potential—we consider a four-vertex process:

1. EM-bound matter couples to scalar field ($\sqrt{\alpha}$)
2. Scalar field couples to gravitational potential ($\sqrt{\alpha}$)
3. Gravitational potential couples to scalar field ($\sqrt{\alpha}$)
4. Scalar field modifies atomic transition frequency ($\sqrt{\alpha}$)

Combined: $(\sqrt{\alpha})^4 = \alpha^2$.

Including a standard loop factor of 2π:

$$k_\alpha = \frac{\alpha^2}{2\pi}. \tag{65}$$

6.3 Status of the heuristic

We present this as a *heuristic* motivating specific powers of α, not as a rigorous derivation. The essential point is that the observed numerical relations are consistent with a vertex-counting structure, and this structure yields falsifiable predictions.

A formal derivation within the full DFD Lagrangian framework is given in the companion paper [16]. The present discussion establishes that the appearance of $\sqrt{\alpha}$ and α^2 is natural from a QED perspective and not numerological coincidence.

7 Universal clock predictions

If $K_A = k_\alpha S_A^\alpha$ with $k_\alpha = \alpha^2/(2\pi)$, every atomic clock has a predicted gravitational coupling:

The prediction is falsifiable: any clock comparison yielding $K_A - K_B \neq k_\alpha(S_A^\alpha - S_B^\alpha)$ would exclude the universal α-coupling hypothesis.

The Cs/Sr channel has $\Delta S^\alpha = 2.77$, among the largest available, amplifying any signal by nearly a factor of 50 compared to channels with $\Delta S^\alpha \sim 0.1$.

7.1 Predicted signal for near-term experiments

For $k_\alpha = \alpha^2/(2\pi)$, the expected annual amplitude in Cs/Sr is

$$|y_{\mathrm{Sr}}^{\mathrm{pred}}| = 3.9 \times 10^{-15}. \tag{66}$$

Species	Transition	S_A^α	K_A^{pred} $(\times 10^{-5})$
^{133}Cs	Hyperfine	2.83	2.40
^{87}Rb	Hyperfine	2.34	1.98
^{1}H	1S-2S	2.00	1.70
^{87}Sr	Optical	0.06	0.05
^{171}Yb$^+$	E2	1.00	0.85
^{171}Yb$^+$	E3	−5.95	−5.04
^{27}Al$^+$	Optical	0.008	0.007
^{199}Hg$^+$	Optical	−2.94	−2.49

Table 1: Predicted gravitational couplings $K_A = k_\alpha S_A^\alpha$ assuming $k_\alpha = \alpha^2/(2\pi) = 8.5 \times 10^{-6}$. Values of S_A^α from Refs. [21–24].

Over a six-month baseline spanning perihelion:

$$\Delta \left(\frac{\nu_{\text{Cs}}}{\nu_{\text{Sr}}} \right) \approx 4 \times 10^{-15}. \tag{67}$$

Modern optical clock comparisons achieve fractional uncertainties of $\sim 10^{-17}$ at one-day averaging [25, 26]. Over a six-month campaign, systematic-limited precision of $\sim 3 \times 10^{-16}$ is achievable.

If the predicted signal is present:

$$\text{Significance} = \frac{4 \times 10^{-15}}{3 \times 10^{-16}} \approx 13\sigma. \tag{68}$$

This would constitute a definitive detection or exclusion of the specific $k_\alpha = \alpha^2/(2\pi)$ hypothesis.

8 Experimental determination of k_a

The k_a parameter controls the strength of scalar self-interaction and thus the size of both astrophysical and laboratory deviations from GR. Determining k_a (or setting bounds on it) therefore requires combining information from multiple regimes.

8.1 Astrophysical constraints

Galaxy rotation curves and their scaling relations can be used to infer an effective acceleration scale a_0^{gal} in the deep low-acceleration regime. [5] In the scalar self-interaction picture, this effective scale is related to k_a and the characteristic density and size of the galaxy by

$$a_0^{\text{gal}} \sim \frac{c^2}{k_a R_{\text{eff}}}. \tag{69}$$

If one adopts a phenomenological value $a_0^{\text{gal}} \approx 1.2 \times 10^{-10}\,\text{m/s}^2$, this provides one handle on k_a for typical disk galaxies of known R_{eff}.

With the α-relation $k_a = 3/(8\alpha) \approx 51.4$, this constraint is automatically satisfied for galaxies with characteristic radii $R_{\text{eff}} \sim 10$ kpc, which is indeed the typical scale for spiral galaxies exhibiting MOND-like behavior.

Cosmological data, on the other hand, constrain the combination $a_\star(\bar{\rho}) \sim cH_0\sqrt{3/(2k_a)}$ discussed above. Requiring that this be of order the observed late-time acceleration implies that k_a must not be extremely small or large; otherwise the scalar self-interaction would either overwhelm or be negligible compared to the ΛCDM fit. [6, 12]

The value $k_a = 3/(8\alpha) \approx 51$ satisfies all these constraints simultaneously.

8.2 Clock-based strategies

Atomic clock experiments provide a complementary and, in some ways, cleaner probe of k_a. The basic strategy is:

1. Choose two clock species A and B with calculable and significantly different sensitivity coefficients K_A and K_B.
2. Deploy clocks at two or more heights separated by a distance Δh in a gravitational field with known density profile $\rho(h)$, such as the Earth's near-surface environment.
3. Measure the fractional ratio shift $\Delta(\nu_A/\nu_B)/(\nu_A/\nu_B)$ as a function of Δh and compare to the GR prediction (which is essentially zero for the ratio).
4. Use Eq. (38) to infer or bound the combination $(K_A - K_B)/\sqrt{k_a}$, and thus k_a once the K_A are known or constrained from atomic theory. [7]

Current and near-future optical lattice clock networks, both ground-based and space-based, already operate at fractional frequency precision better than 10^{-17}–10^{-18}. [9, 10] This is sufficient to probe extraordinarily small deviations from LPI over height differences of order 10^2–10^3 m, especially when multiple species are compared.

8.3 Consistency with existing tests

Any scalar self-interaction model must remain consistent with the impressive null tests of the equivalence principle and GR obtained from experiments such as MICROSCOPE, binary pulsar timing, and the gravitational wave observations of LIGO and Virgo. [8, 11, 13–15] In the present framework, this translates into bounds on k_a and the products $k_a K_A$.

The essential point is that the same k_a enters all three regimes we have discussed:

- galaxy dynamics (through a_0^{gal}),
- cosmic acceleration (through $a_\star(\bar{\rho})$),
- clock tests (through the ratio shifts in Eq. (38)).

This eliminates the freedom to tune each sector independently and turns what might otherwise be a collection of unrelated anomalies into a network of cross-checks. Any choice of k_a that fits galaxies but grossly violates clock or GW constraints, or vice versa, is ruled out.

9 Limitations, strong fields, and gravitational waves

The analysis in this paper is intentionally restricted to the weak-field, quasi-static sector of a scalar refractive-index theory, where a single scalar field ψ and its gradient determine the effective potential and test-mass acceleration. In this limit, the relevant invariant is $a^2 \equiv \boldsymbol{a} \cdot \boldsymbol{a}$, and the self-interaction parameter k_a fixes how departures from Newtonian gravity emerge in low-acceleration environments. All of the results above are derived in this regime: static or slowly varying configurations, no strong-field horizons, and no explicit radiation sector.

From the broader DFD point of view, this is a controlled truncation rather than a complete theory. In the full framework [16], the refractive field ψ fixes an optical metric $g_{\mu\nu}[\psi]$, and a separate transverse-traceless radiation sector can be added consistently, yielding tensor gravitational waves with the observed polarizations and near-luminal propagation speed. The strong-field structure of that completion, and its confrontation with LIGO/Virgo events and horizon-scale tests, are treated in the companion DFD analysis.

The present work deliberately does *not* re-derive or fit the tensor gravitational-wave sector. In particular, we do not attempt to:

- compute full inspiral-merger-ringdown waveforms in the k_a-deformed theory;
- revisit polarization constraints from LIGO/Virgo beyond the requirement that a viable completion retain a transverse-traceless sector;
- analyse strongly curved, dynamical spacetimes where higher-order invariants or additional fields may become important.

Within its narrow scope, the contribution of this paper is therefore precise: it isolates a single acceleration-squared invariant a^2 and shows how a scalar self-interaction governed by k_a can link galaxies, cosmology, and clocks in the weak-field, quasi-static regime—and demonstrates that the resulting parameters satisfy striking numerical relations involving the fine-structure constant α.

10 Implications for the DFD program

Within the broader Density Field Dynamics program [16], the central idea is that a single scalar density or refractive field controls both the effective metric for light and matter and the stochastic structure of quantum measurement. Those aspects lie beyond the scope of this paper, which has focused solely on the classical weak-field gravity sector.

Nevertheless, the emergence of a universal acceleration-squared invariant a^2 with self-coupling k_a, together with the α-relations derived in Section 5, has several important implications:

1. It provides a simple and robust organizing principle: everywhere the scalar field has a gradient, there is an associated local scale $a_\star(\rho)$ set by Eq. (13). Physical phenomena as diverse as galaxy rotation curves, cosmic acceleration, and clock redshifts are then different windows into this same scalar gradient energy.
2. It sharply reduces the number of genuinely free parameters in the gravitational sector. Once α and H_0 are specified, the relations in Section 5 fix k_a, k_α, and a_0 completely. The others become predictions rather than independent fits.

3. It suggests a natural hierarchy of regimes. High-acceleration systems such as the Solar System lie firmly in the linear regime $\nabla \cdot \boldsymbol{a} \gg k_a a^2/c^2$, reproducing GR and Newtonian gravity to high accuracy. Low-acceleration, low-density environments lie in the nonlinear regime $k_a a^2/c^2 \gtrsim \nabla \cdot \boldsymbol{a}$, where MOND-like and dark-energy-like phenomena emerge.

4. It provides a clean target for both theoretical and experimental work: the precise determination of $k_\alpha = \alpha^2/(2\pi)$ through clock experiments would confirm the α-gravity connection and validate the entire DFD structure.

11 Conclusions and outlook

We have identified and analyzed a simple but powerful structural feature of scalar refractive-index theories of gravity: a universal acceleration-squared invariant $a^2 = \boldsymbol{a}\cdot\boldsymbol{a}$ that appears linearly in the field equation with a dimensionless self-coupling k_a/c^2. This leads naturally to a density-dependent acceleration scale $a_\star(\rho)$ that:

- produces MOND-like scaling in galaxies without introducing an arbitrary new constant unrelated to the density environment;
- matches the order of magnitude of the cosmic acceleration scale when ρ is taken to be the mean cosmic density;
- directly controls composition-dependent gravitational redshift effects for atomic clocks via species-dependent couplings K_A.

Most strikingly, the phenomenological parameters governing this structure satisfy simple numerical relations involving the fine-structure constant:

$$a_0 = 2\sqrt{\alpha}\, cH_0 \quad \text{(within } H_0 \text{ uncertainty)}, \tag{70}$$

$$k_\alpha = \frac{\alpha^2}{2\pi} \quad \text{(consistent with data at } \sim 2\sigma\text{)}, \tag{71}$$

$$k_a = \frac{3}{8\alpha}. \tag{72}$$

These relations contain no free parameters beyond α and H_0. A vertex-counting heuristic motivates the appearance of $\sqrt{\alpha}$ (two vertices) and α^2 (four vertices), connecting MOND phenomenology to atomic clock physics through the fine-structure constant.

The main conceptual achievement is that a single structural parameter k_a—together with the invariant a^2 and its connection to α—links three previously disparate acceleration scales: galactic a_0, cosmic a_Λ, and laboratory-scale sensitivities in precision metrology. This closes a loop in the gravitational sector of the Density Field Dynamics program: once α and H_0 are specified, the others are no longer free to vary independently.

From an experimental perspective, the most promising near-term probes of the α-relations are multi-species atomic clock networks, which can measure or bound composition-dependent gravitational redshift at levels far beyond what is accessible to astrophysical observations alone. The prediction $k_\alpha = \alpha^2/(2\pi) \approx 8.5 \times 10^{-6}$ can be tested at $> 10\sigma$ precision by ongoing and planned optical clock campaigns. If confirmed, this would establish a direct link between the fine-structure constant and gravitational phenomenology—a connection uniquely predicted by the DFD framework.

Acknowledgements

The author thanks the many experimental teams developing ever more precise optical clocks, without which the connection between fundamental gravity and metrology would remain purely theoretical.

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