WELL-POSEDNESS AND BOUNDARY VALUE PROBLEMS FOR A CLASS OF QUASILINEAR DIVERGENCE-FORM EQUATIONS ARISING IN DENSITY FIELD DYNAMICS

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Abstract. We study the quasilinear elliptic partial differential equation

$$-\nabla \cdot \big(\mu(|\nabla\psi|)\nabla\psi\big) = f \quad \text{in } \Omega \subseteq \mathbb{R}^3,$$

where μ is a nonlinear constitutive function. Motivated by density-field models of gravitational optics, we develop a rigorous framework for existence, uniqueness, and regularity of weak solutions, extend the analysis to exterior domains with asymptotically flat boundary conditions, and incorporate monotone nonlinear Robin–Neumann conditions modeling photon-spheres and horizons. We further establish stability estimates, continuous dependence on data, and parabolic well-posedness using nonlinear semigroup theory. A variational formulation, catalog of admissible μ-families, and finite element method (FEM) implementation outline are provided. Open problems relevant to global existence and singularity formation are discussed.

Contents

1. Introduction

We investigate the nonlinear elliptic equation

$$-\nabla \cdot \big(\mu(|\nabla\psi|)\nabla\psi\big) = f, \tag{1}$$

posed on a domain $\Omega \subseteq \mathbb{R}^3$. Here $\psi : \Omega \to \mathbb{R}$ is the unknown scalar potential, $\mu : [0,\infty) \to (0,\infty)$ is a nonlinear coefficient, and f represents a source term. Such equations belong to the class of *quasilinear divergence-form PDEs* with p-growth, generalizing the p-Laplacian. They arise in fluid

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mechanics, nonlinear diffusion, and, in recent physical models, as optical potentials in effective theories of gravitation.

Notation.

- $L^p(\Omega)$: standard Lebesgue spaces, $1 \le p \le \infty$.
- $W^{1,p}(\Omega)$: Sobolev space of L^p functions with L^p weak derivatives.
- $V := W_0^{1,p}(\Omega)$: closure of $C_c^\infty(\Omega)$ in $W^{1,p}$.
- V': dual of V.
- $\langle \cdot, \cdot \rangle$: duality pairing between V' and V.

1.1. **Physical motivation for μ and boundary conditions.** In density-field models of gravitation, one introduces an "optical potential" ψ such that the refractive index is $n = e^\psi$. The flux coefficient $\mu(|\nabla\psi|)$ encodes the response of the medium to spatial gradients of ψ. Its form determines how weak-field Newtonian gravity, strong-field photon spheres, and effective horizon behavior emerge.

Boundary conditions are motivated as follows:

- **Photon sphere:** defined by an extremum of the optical circumference $n(r)r$. This yields a Robin-type condition with coefficient $\kappa_{\text{opt}}(\psi)$ tied to the local optical speed.
- **Horizon:** at the surface where outgoing null characteristics stall, one enforces an "ingoing flux only" condition. Mathematically this corresponds to a nonlinear Neumann condition eliminating outgoing flux. We emphasize this is *physically motivated but mathematically non-standard*, and justifying it within elliptic PDE theory is an open problem.

2. Assumptions on μ

We assume $\mu : [0, \infty) \to (0, \infty)$ satisfies:

- **(A1) Continuity:** μ is continuous on $[0, \infty)$.
- **(A2) Coercivity:** $\exists \alpha > 0$, $p \ge 2$ such that
$$\mu(|\xi|)|\xi|^2 \ge \alpha|\xi|^p \quad \forall \xi \in \mathbb{R}^3.$$
- **(A3) Growth:** $\exists \beta > 0$ such that
$$|\mu(|\xi|)\xi| \le \beta(1 + |\xi|)^{p-1}.$$
- **(A4) Monotonicity:** For all $\xi, \eta \in \mathbb{R}^3$,
$$\big(\mu(|\xi|)\xi - \mu(|\eta|)\eta\big) \cdot (\xi - \eta) \ge 0.$$
If strict, uniqueness follows.

Examples include the p-Laplacian $\mu(s) = s^{p-2}$, saturating nonlinearities $\mu(s) = (1 + s^2)^{(p-2)/2}$, and MOND-like regularized forms $\mu(s) = s/\sqrt{s^2 + s_a^2}$ [6, 7].

3. Weak formulation and variational structure

Define the flux map $a(\xi) := \mu(|\xi|)\xi$. For $\psi \in W^{1,p}(\Omega)$ with boundary data $\psi = \psi_D$, the weak formulation is:
$$\int_\Omega a(\nabla\psi) \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2}$$

Define the energy density
$$H(\xi) := \int_0^1 a(t\xi) \cdot \xi \, dt,$$
so that $a(\xi) = \nabla_\xi H(\xi)$. Then the functional
$$\mathcal{E}[\psi] := \int_\Omega H(\nabla\psi) \, dx - \int_\Omega f\psi \, dx$$

is convex and coercive under (A1)–(A3).

4. Main results

Theorem 4.1 (Existence). *Under (A1)–(A4), for any $f \in V'$, there exists a weak solution $\psi \in W^{1,p}(\Omega)$ of* (1) *attaining the prescribed boundary data.*

Theorem 4.2 (Uniqueness). *If $a(\xi) = \mu(|\xi|)\xi$ is strictly monotone, the weak solution of Theorem 4.1 is unique.*

Theorem 4.3 (Regularity). *If $f \in L^q(\Omega)$ with $q > 3/p'$, then any weak solution ψ is locally Hölder continuous: $\psi \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. If $\mu \in C^1$ and $f \in C^{0,\gamma}$, then $\psi \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

Proofs follow standard methods from monotone operator theory and quasilinear elliptic regularity [1, 2, 3, 4].

5. Exterior domains and optical boundary conditions

Let $\Omega = \mathbb{R}^3 \setminus \overline{B_R}$ denote an exterior domain. We impose:

- **Asymptotic flatness:** $\psi(x) \to 0$ as $|x| \to \infty$.
- **Photon-sphere boundary:** Nonlinear Robin condition

  $$a(\nabla\psi) \cdot n + \kappa_{\mathrm{opt}}(\psi)\,\psi = g_{\mathrm{ph}} \quad \text{on } \Gamma_{\mathrm{ph}},$$

  with κ_{opt} positive and bounded.
- **Horizon boundary:** Ingoing-flux Neumann condition

  $$a(\nabla\psi) \cdot n = g_{\mathrm{hor}}, \quad \text{with outgoing flux set to zero.}$$

  This asymmetric boundary condition is physically motivated but not standard in elliptic PDE theory. A full mathematical justification remains open.

Theorem 5.1 (Exterior well-posedness). *Under (A1)–(A4) and the above boundary conditions, there exists a weak solution $\psi \in W^{1,p}(\Omega)$. If the boundary operators are strictly monotone, the solution is unique.*

6. Stability and continuous dependence

Theorem 6.1 (Stability). *Let ψ_1, ψ_2 be solutions with data (f_1, BC_1), (f_2, BC_2). If a is strongly monotone and locally Lipschitz, then*

$$\|\nabla(\psi_1 - \psi_2)\|_{L^p(\Omega)} \le C\Big(\|f_1 - f_2\|_{V'} + \|\mathrm{BC}_1 - \mathrm{BC}_2\|\Big).$$

7. Parabolic extension and semigroup theory

Consider

$$\partial_t\psi - \nabla \cdot \big(\mu(|\nabla\psi|)\nabla\psi\big) = f(t,x).$$

Let $A : V \to V'$ be the monotone operator $A(\psi) = -\nabla \cdot a(\nabla\psi)$. By Crandall–Liggett theory [5], $-A$ generates a contraction semigroup on $L^2(\Omega)$.

Theorem 7.1 (Parabolic well-posedness). *Under (A1)–(A4), there exists a unique evolution $\psi \in L^p(0,T; W^{1,p}(\Omega)) \cap C([0,T]; L^2(\Omega))$. If f is time-independent and boundary operators are dissipative, then solutions converge to a steady state as $t \to \infty$.*

8. Finite element method (FEM) implementation

The weak form (2) is directly implementable in finite element packages. Nonlinear terms are treated via Newton iteration with Jacobian

$$A_{ij}(\nabla\psi) = \mu(|\nabla\psi|)\delta_{ij} + \mu'(|\nabla\psi|)\frac{\partial_i\psi\,\partial_j\psi}{|\nabla\psi|}.$$

Remark 8.1. At $|\nabla\psi| \to 0$, the Jacobian may become ill-conditioned. A practical remedy is to replace $|\nabla\psi|$ by $\sqrt{|\nabla\psi|^2 + s_0^2}$ with small $s_0 > 0$ (regularization). For background on FEM analysis of quasilinear PDEs, see [8, 9].

Optical boundary conditions appear as Robin/Neumann integrals in the variational form.

9. Catalog of admissible μ-families

- **p-Laplacian:** $\mu(s) = s^{p-2}$.
- **Saturating:** $\mu(s) = (1 + s^2)^{(p-2)/2}$.
- **Regularized MOND-like:** $\mu(s) = \frac{s}{\sqrt{s^2+s_a^2}}$ [6, 7].
- **Anisotropic:** μ replaced by positive-definite tensor $M(\nabla\psi)$.

10. Open problems

- Global existence with physically realistic sources f.
- Gradient blow-up and singularity formation.
- Regularity near horizons under nonlinear asymmetric BCs.
- Mathematical justification of the "ingoing flux only" horizon condition.
- Coupling of the scalar ψ-equation to tensorial sectors in relativistic completions.

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Figure: Exterior Domain with Optical Boundaries