

# General Relativity as the Padé Approximant of Density Field Dynamics: Refractive gravity on flat space and its relation to Schwarzschild geometry

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We show that general relativity arises as the simplest rational truncation of Density Field Dynamics (DFD) in the gravitational clock-rate sector. Define the lapse-squared scalar  $\mathcal{L}(u) \equiv c^2/|g_{tt}|$ , a clock-rate / redshift observable, with  $u \equiv GM/(\varrho c^2)$ . The isotropic-coordinate Schwarzschild form of GR gives  $\mathcal{L}_{\text{GR}}(u) = [(1+u/2)/(1-u/2)]^2$ , while DFD's exterior solution gives  $\mathcal{L}_{\text{DFD}}(u) = \exp(2u)$ . The exact identity

$$\mathcal{L}_{\text{GR}}(u) = [P_{1,1}(\exp(u))]^2$$

extends to a Padé hierarchy in which  $[P_{m,m}(\exp(u))]^2 = \exp(2u) + O(u^{2m+1})$  for every  $m \geq 1$ , with each finite- $m$  truncation carrying a Padé pole that recedes to infinity only as  $m \rightarrow \infty$ . **GR is the  $m = 1$  slot; DFD is the entire-function limit.** The Schwarzschild horizon at  $r = 2GM/c^2$  is the Padé pole of the  $m = 1$  truncation; DFD's exponential has no finite pole and its  $\mu \rightarrow 1$  exterior  $\psi(r) = 2GM/(c^2 r)$  is everywhere finite, with  $r = 2GM/c^2$  appearing as a photon sphere rather than a horizon. The two theories agree through  $O(u^2)$  by construction — consistent with all gravitational-redshift and clock observations to date and reproducing the post-Newtonian parameter  $\beta = 1$  — and first differ at  $O(u^3)$ , generating a  $\sim 4.6\%$  larger black-hole shadow. The identity is in the lapse-squared scalar, which controls clock rates, redshift, the Newtonian limit,  $\beta$ , and horizon structure; the spatial metric, ray-optics, and the PPN parameter  $\gamma$  are not its consequences and are established separately for DFD via the physical metric in [8]. Several prior constructions also produce exponential lapses (Papapetrou 1954, Yilmaz 1958, Dicke 1957, Puthoff polarizable vacuum 1999/2002, Broekaert 2008), so the Padé identity itself holds for any of them; what makes the present statement an inter-theory reduction rather than a tautology is that DFD is not embedded inside GR. The Yilmaz exponential, for example, is itself a GR solution interpretable as a wormhole with exotic matter, whereas DFD's flat- $\mathbb{R}^3$  elliptic field equation admits no throat and requires no exotic matter. Current Event Horizon Telescope data on M87\* and Sgr A\* are consistent with both GR and DFD at the present precision; the predicted  $\sim 4.6\%$  shadow excess is the proximal observational discriminator.

## I. INTRODUCTION

The observational equivalence of general relativity (GR) and competing metric theories at post-Newtonian order is well-established [1, 2]. Within the *refractive* tradition of gravity, a variety of constructions — from Eddington's 1920 optical-medium analogy [3] to Dicke's 1957 variable-speed-of-light formulation [4] to Puthoff's polarizable vacuum [5, 6] and Broekaert's scalar analog formulation [7] — produce weak-field refractive indices that match GR through at least linear order in the gravitational potential. The exponential form  $n = \exp(2GM/rc^2)$  has appeared multiple times in this literature. The purpose of this paper is to observe a specific mathematical identity between GR's isotropic-coordinate Schwarzschild lapse-squared and the exponential lapse-squared of Density Field Dynamics (DFD), and to analyze its structural content.

## A. Density Field Dynamics

DFD [8] is formulated on flat three-dimensional Euclidean space  $\mathbb{R}^3$  with time as an external parameter. The fundamental object is a dimensionless scalar field  $\psi(\mathbf{x}, t)$  called the loading. The optical refractive index is

$$n_{\text{DFD}}(\psi) = e^\psi, \quad (1)$$

and matter acceleration is

$$\mathbf{a} = \frac{1}{2} c^2 \nabla \psi. \quad (2)$$

The static field equation is a nonlinear elliptic PDE on  $\mathbb{R}^3$ :

$$\nabla \cdot [\mu(|\nabla \psi|/a_*) \nabla \psi] = -\frac{8\pi G}{c^2} \rho_m, \quad (3)$$

with  $\mu$  an interpolation function,  $a_*$  an acceleration scale, and  $\rho_m$  ordinary matter density [8]. Equation (3) is structurally the AQUAL field equation of Bekenstein–Milgrom modified-Newtonian-dynamics [11–13], with  $a_*$  identified with Milgrom's  $a_0$ . It is not Einstein's equation: the geometry of  $\mathbb{R}^3$  is flat and fixed, there is no Einstein tensor, no Hilbert action, no dynamical 4-metric.

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## B. The exponential form

The exponential form (1) is postulated on the basis of a multiplicative composition axiom for refractive loading: when two loading fields  $\psi_1, \psi_2$  are superposed, their refractive effects multiply,  $n(\psi_1 + \psi_2) = n(\psi_1)n(\psi_2)$ . Under continuity and measurability, Cauchy's functional equation implies  $n(\psi) = e^{k\psi}$  for some constant  $k$  [9]. The factor of 2 in  $n_{\text{DFD}}(u) = \exp(2u)$  below is fixed separately by normalization against observed gravitational light bending.

Multiplicative composition is a *DFD-specific constitutive postulate*, not a universal feature of optical media. Series media compose additively via optical path length, and effective-medium theories (Maxwell–Garnett, Bruggeman) compose permittivity  $\epsilon$  by volume averaging rather than multiplying  $n$  [10]. The DFD postulate is well-defined and testable but not inherited from classical optics; the Padé identity below does not depend on its derivation.

## C. Relation to prior refractive-gravity work

The exponential refractive form has a substantial prior history. We distinguish DFD from each principal precedent before stating our result.

*Yilmaz exponential metric* [23, 24] and later work [25–29]. This is a four-dimensional Lorentzian metric  $ds^2 = -e^{-2m/r}c^2dt^2 + e^{+2m/r}(dr^2 + r^2d\Omega^2)$ , interpreted as a solution of Einstein–Klein–Gordon equations with an antiscalar source. It is a solution *within* GR, not outside it. At the optical level its radial refractive index is  $n_{\text{Yilmaz}} = e^{2m/r}$ , matching DFD's functional form; the distinction is entirely ontological (see Sec. VI).

*Dicke variable-speed-of-light* [4]. Proposes a position-dependent index of refraction in otherwise flat spacetime, reproducing weak-field GR predictions through PPN order. Differs from DFD in having no field equation analogous to (3) and in taking the speed-of-light variation as primitive rather than as a derived consequence of a scalar loading.

*Puthoff polarizable vacuum* [5, 6]. Introduces a vacuum dielectric function  $K = \exp(2GM/rc^2)$  that modifies speed, frequency, and ruler scales simultaneously, on a flat background. Closest prior art to DFD in that both are flat-background exponential refractive formulations. Distinguished from DFD by (a) derivation route: PV from a scaled-ruler/scaled-clock heuristic, DFD from Cauchy composition on  $\psi$ ; (b) field equation: PV postulates the exponential directly, DFD has the nonlinear elliptic (3) with MOND-type interpolation  $\mu$ ; (c) cosmological structure: DFD is embedded in a specific topological framework on  $\mathbb{C}P^2 \times S^3$  yielding Standard-Model parameters and  $\alpha^{-1}$  from Chern–Simons level quantization [8].

*Broekaert scalar analog* [7]. Derives refractive formulations of gravity from variational principles on flat

backgrounds; specifically addresses PPN equivalence. Broekaert's construction is Lagrangian-first, DFD's is postulate-first via the composition axiom.

*Optical-metric tradition*. Gordon [14] introduced the optical metric  $\tilde{g}_{\mu\nu} = g_{\mu\nu} + (1 - n^{-2})u_\mu u_\nu$  for light propagation in moving or refractive media within GR. Plebanski [15] developed the analogy between curved spacetime and optical media. De Felice [16] and Perlick [17] consolidated these ideas; Ye and Lin [18] derived the specific exponential weak-field form from GR in the optical-medium analogy. DFD's derived optical metric  $\tilde{g}_{\mu\nu} = \text{diag}(-c^2/n^2, 1, 1, 1)$  is conformally related to Gordon's form with a flat reference metric, and serves the same bookkeeping purpose. The distinction is that in the Gordon–Plebanski tradition the optical metric is *derived from* a curved GR spacetime, whereas in DFD the optical metric is a derived object on a fundamentally flat  $\mathbb{R}^3$ , with  $\psi$  as primitive.

*Analog gravity* [20–22]. A related but distinct program in which emergent causal structure (horizons, trapped surfaces) arises on a fundamentally flat substrate from matter flow or refractive-index variation. The analog-gravity precedent is important to DFD's claim (Sec. V) that a flat substrate admits no wormhole throat: such a claim must be made explicitly about the spatial 3-metric, not merely about the existence of a flat substrate, because analog constructions demonstrate that emergent causal structure can exist on flat backgrounds.

## D. What this paper does

We observe a specific Padé-approximation identity between GR's squared inverse-lapse and DFD's exponential lapse-squared (Sec. II), analyze the order-by-order series agreement (Sec. III) and the strong-field divergence (Sec. IV), distinguish the DFD ontology from Yilmaz-type GR solutions (Secs. V–VI), and identify observational discriminants (Sec. VII). The Padé identity as a named relation is, to our knowledge, novel; the exponential form itself is not.

# II. THE PADÉ IDENTITY

## A. DFD's lapse-squared and refractive index

Around a spherically symmetric mass  $M$ , DFD's exterior solution in the  $\mu \rightarrow 1$  regime gives  $\psi(r) = 2GM/(c^2r)$ . With  $u \equiv GM/(\rho c^2)$ , where  $\rho$  denotes the isotropic radial coordinate in GR and the flat-space radial coordinate in DFD, we have  $\psi = 2u$  on the exterior.

DFD's matter-coupling (physical) metric has  $g_{tt} = -c^2e^{-\psi}$  [8], so the lapse-squared scalar is

$$\mathcal{L}_{\text{DFD}}(u) \equiv \frac{c^2}{|g_{tt}|} = e^\psi = \exp(2u). \quad (4)$$

This is the redshift / clock-rate scalar of DFD's matter sector and the natural counterpart, in DFD, of GR's lapse-squared. Independently, by Postulate P1 the optical refractive index is  $n_{\text{DFD}}(u) = e^\psi = \exp(2u)$ , governing light propagation through the optical metric  $d\tilde{s}^2 = -c^2 dt^2/n^2 + d\mathbf{x}^2$ . The lapse-squared  $\mathcal{L}_{\text{DFD}}$  and the refractive index  $n_{\text{DFD}}$  are *distinct physical quantities arising from different metrics in DFD's two-metric structure*, but they coincide as functions of  $u$  on the exterior solution:  $\mathcal{L}_{\text{DFD}}(u) = n_{\text{DFD}}(u) = \exp(2u)$ . The Padé identity below operates on  $\mathcal{L}_{\text{DFD}}$  (the lapse-squared); where Sec. IV C uses  $n(r) = \exp(2GM/(c^2 r))$ , that refers to the refractive index by P1, which on the exterior takes the same numerical value.

### B. Padé approximant of exp

The  $[1, 1]$  Padé approximant of  $\exp(x)$  about  $x = 0$  is the unique rational function whose Taylor expansion matches that of  $\exp$  through order  $x^2$ :

$$P_{1,1}(\exp(x)) = \frac{1 + x/2}{1 - x/2}. \quad (5)$$

### C. Isotropic Schwarzschild: scalar conventions

The Schwarzschild exterior in isotropic radial coordinate  $\varrho$  reads

$$ds^2 = -A^2(\varrho) c^2 dt^2 + B^2(\varrho) [d\varrho^2 + \varrho^2 d\Omega^2], \quad (6)$$

with

$$A(\varrho) = \frac{1 - u/2}{1 + u/2}, \quad B(\varrho) = \left(1 + \frac{u}{2}\right)^2. \quad (7)$$

Several distinct refractive-index conventions are in use in the literature; each corresponds to a different observable.

- *Gordon ray-optics convention:*  $n_{\text{ray}} = B/A$ , governing spatial ray-bending and the direction of photon propagation in a 3+1 split. This is the natural convention in the Gordon–Plebanski optical-metric tradition [14, 15, 17].
- *Inverse-lapse convention:*  $n = 1/\sqrt{|g_{tt}|/(-c^2)} = 1/A$ , governing clock rates and gravitational redshift. Used in phase-based PPN analyses [16, 19].
- *Squared inverse-lapse:*  $n^2 = 1/A^2$ , which appears in phase-speed-squared expressions for gravitational-redshift tests and in the PPN expansion of the  $tt$  metric component.

The Padé identity below uses the squared inverse-lapse scalar,

$$1/A^2(u) \equiv \frac{c^2}{|g_{tt}|} = \left(\frac{1 + u/2}{1 - u/2}\right)^2, \quad (8)$$

which governs the gravitational redshift (clock frequency ratios) and fixes the PPN parameter  $\beta$  through the  $O(u^2)$  coefficient of  $g_{tt}$ . By contrast, the PPN parameter  $\gamma$  lives in the spatial metric  $g_{ij}$  and is not determined by the lapse alone. Under this lapse-sector convention, DFD's  $\mathcal{L}_{\text{DFD}}(u) = \exp(2u)$  and GR's  $\mathcal{L}_{\text{GR}}(u) = [(1 + u/2)/(1 - u/2)]^2$  are directly comparable as clock-rate ratios.

*a. Note on DFD notation.* A reader may worry about an apparent collision between “ $n = e^\psi$ ” as DFD's optical refractive index (Postulate P1) and the lapse-squared scalar  $\mathcal{L} = 1/A^2$  used in this paper. In DFD these are distinct physical quantities arising from the theory's two-metric structure: the optical metric  $d\tilde{s}^2 = -c^2 dt^2/n^2 + d\mathbf{x}^2$  governs light (Postulate P1, refractive index  $n = e^\psi$ ), while the matter-coupling physical metric has  $g_{tt} = -c^2 e^{-\psi}$ , giving lapse-squared  $\mathcal{L} = c^2/|g_{tt}| = e^\psi$ . On the spherically symmetric exterior  $\psi = 2u$ , both quantities take the same numerical form  $\exp(2u)$ , but they are not the same physical object:  $n$  controls light propagation,  $\mathcal{L}$  controls clock rates. The Padé identity in this paper is a statement about  $\mathcal{L}$ , the clock-rate scalar; where Section IV C uses  $n(r) = \exp(2GM/(c^2 r))$  in describing the DFD exterior, that is the optical refractive index by P1, which on the exterior coincides numerically with  $\mathcal{L}$ .

Under the Gordon ray-optics convention the GR refractive index for radial null geodesics is  $B/A = (1 + u/2)^3/(1 - u/2)$ , whose Taylor series is  $1 + 2u + \frac{7}{4}u^2 + \dots$ ; this disagrees with DFD's  $\exp(2u)$  at  $O(u^2)$  rather than  $O(u^3)$ . The Padé identity we establish is specifically a lapse-sector (clock-rate/redshift) identity; it is *not* a ray-optics identity, and it does not by itself establish agreement of light-bending or Shapiro-delay observables, which involve the spatial metric. Full PPN agreement between DFD and GR, including  $\gamma = 1$ , is established separately using DFD's physical metric in [8] § PPN and is *not* a consequence of the present identity.

### D. The identity

$$\boxed{\mathcal{L}_{\text{GR}}(u) = [P_{1,1}(\exp(u))]^2}. \quad (9)$$

The lapse-squared scalar of isotropic-coordinate Schwarzschild is exactly the square of the  $[1, 1]$  Padé approximant of DFD's exponential. Figure 1 shows the two functions and their relative difference.

## III. ORDER-BY-ORDER COMPARISON

Taylor expanding both forms about  $u = 0$ :

$$\mathcal{L}_{\text{DFD}}(u) = 1 + 2u + 2u^2 + \frac{4}{3}u^3 + \frac{2}{3}u^4 + \dots, \quad (10)$$

$$\mathcal{L}_{\text{GR}}(u) = 1 + 2u + 2u^2 + \frac{3}{2}u^3 + u^4 + \dots. \quad (11)$$

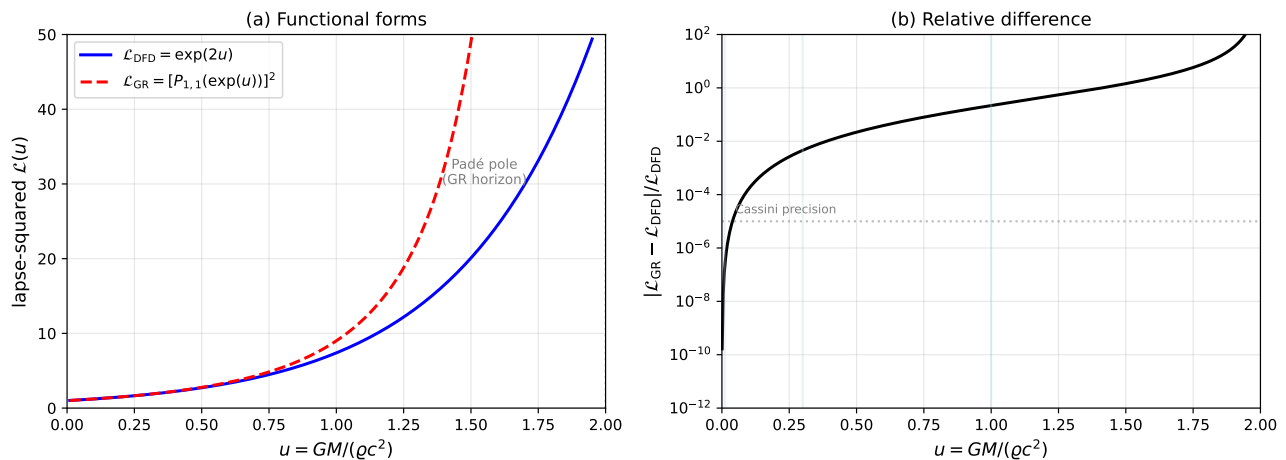


FIG. 1. (a) DFD’s  $\mathcal{L}_{\text{DFD}}(u) = \exp(2u)$  (blue) and GR’s  $\mathcal{L}_{\text{GR}}(u) = [P_{1,1}(\exp(u))]^2$  (red dashed). The GR form has a pole at  $u = 2$  corresponding to the Schwarzschild horizon coordinate; DFD has no pole. (b) Relative difference  $|\mathcal{L}_{\text{GR}} - \mathcal{L}_{\text{DFD}}|/\mathcal{L}_{\text{DFD}}$  on a logarithmic scale. Solar-system tests probe  $u \sim 10^{-6}$ , where the difference is  $\sim 10^{-19}$ ; neutron-star envelopes probe  $u \sim 0.1$ ; EHT shadows  $u \sim 0.3$ .

TABLE I. Taylor coefficients of  $\mathcal{L}_{\text{DFD}}$  and  $\mathcal{L}_{\text{GR}}$  through  $O(u^5)$ .

order $n$	$[u^n]\mathcal{L}_{\text{DFD}}$	$[u^n]\mathcal{L}_{\text{GR}}$	difference
0	1	1	0
1	2	2	0
2	2	2	0
3	4/3	3/2	-1/6
4	2/3	1	-1/3
5	4/15	5/8	-43/120

TABLE II. Relative difference between GR and DFD lapse-squared scalars  $\mathcal{L}(u)$ .

$u$	relative difference	system scale
$10^{-6}$	$\sim 10^{-19}$	solar surface
$10^{-2}$	$\sim 10^{-7}$	compact-star exterior
$10^{-1}$	$\sim 10^{-4}$	neutron-star envelope
0.30	$4.6 \times 10^{-3}$	photon-sphere region
0.50	$2.2 \times 10^{-2}$	Schwarzschild radius scale
1.00	$2.2 \times 10^{-1}$	inside Schwarzschild radius
$\rightarrow 2$	$\rightarrow \infty$	Padé pole / GR horizon

The series agree through  $O(u^2)$  and first diverge at  $O(u^3)$  with coefficient difference 1/6. The  $O(u)$  and  $O(u^2)$  coefficients of  $\mathcal{L} = 1/A^2$  are set by the PPN expansion of  $g_{tt}$ , from which we read  $\beta = 1$  for both theories. The PPN parameter  $\gamma$  is determined by the spatial metric and is *not* accessible from the lapse sector alone; for DFD,  $\gamma = 1$  follows from the physical metric  $g_{ij} = e^{+\psi}\delta_{ij}$  and is established in [8] § PPN, independently of the present identity. Experimentally,  $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$  from Cassini [33] and  $\beta - 1 = (-4.5 \pm 5.6) \times 10^{-5}$  from Hofmann-Müller [34] lunar laser ranging are both consistent with  $\gamma = \beta = 1$ .

As a Lorentz-invariant conservative scalar theory on flat  $\mathbb{R}^3$ , DFD also predicts the preferred-frame and preferred-location parameters  $\alpha_1 = \alpha_2 = \alpha_3 = \xi = 0$  and conservation parameters  $\zeta_{1,2,3,4} = 0$ , matching all ten-parameter PPN constraints [2] identically with GR at post-Newtonian order. The detailed full-PPN derivation is given in [8].

## IV. STRONG-FIELD BEHAVIOR

### A. Numerical divergence

Table II lists representative relative discrepancies  $|\mathcal{L}_{\text{GR}} - \mathcal{L}_{\text{DFD}}|/\mathcal{L}_{\text{DFD}}$ .

### B. The horizon as Padé pole

The Padé approximant has a simple pole at  $u = 2$ . In isotropic coordinates this is  $\varrho = GM/(2c^2)$ ; the isotropic-to-Schwarzschild transformation  $r = \varrho(1 + u/2)^2$  maps to  $r = 2GM/c^2$ , the Schwarzschild event horizon radius. The function  $\exp(2u)$  is entire; at  $u = 2$  it takes the finite value  $e^4 \approx 54.6$ .

*a. Coordinate hygiene.* The Padé comparison above uses GR’s isotropic radial coordinate  $\varrho$ . The explicit DFD exterior solution below is written in the flat-space radial coordinate  $r$  of Euclidean  $\mathbb{R}^3$ . The GR horizon at  $r = 2GM/c^2$  (Schwarzschild), the Padé pole at

$\varrho = GM/(2c^2)$  (isotropic GR), and the DFD photon sphere at  $r = 2GM/c^2$  (flat  $\mathbb{R}^3$ ) all sit at the same physical mass scale  $GM/c^2$  but in three distinct coordinate charts.

### C. DFD's exterior solution has no finite-radius horizon

In the strong-field  $\mu \rightarrow 1$  regime, the vacuum field equation (3) around a spherically symmetric mass  $M$  integrates to [8]

$$\psi(r) = \frac{2GM}{c^2 r}, \quad n(r) = e^{2GM/(c^2 r)}, \quad (12)$$

finite for every  $r > 0$  and divergent only at  $r = 0$ . The local phase speed  $c/n(r)$  is positive for all finite  $r > 0$ ; no finite-radius surface traps light. Applicability of the  $\mu \rightarrow 1$  approximation at the photon-sphere scale is controlled by  $a_*/a_{\text{ph}} \sim 10^{-13}$  for M87\*-class supermassive black holes (and smaller yet for stellar-mass); sub- $\mu \rightarrow 1$  corrections are negligible at astrophysical precision.

The radius  $r = 2GM/c^2$  appears in the DFD exterior as the *photon sphere*, determined by the orbital condition  $d[n(r)r]/dr = 0$ :

$$\frac{d}{dr} \left[ e^{2GM/(c^2 r)} r \right] = 0 \implies r_{\text{ph}}^{\text{DFD}} = \frac{2GM}{c^2}. \quad (13)$$

This is a surface of unstable circular photon orbits, not a causal boundary. The critical impact parameter is  $b_{\text{crit}}^{\text{DFD}} = 2eGM/c^2 \approx 5.44 GM/c^2$ , vs.  $b_{\text{crit}}^{\text{GR}} = 3\sqrt{3}GM/c^2 \approx 5.20 GM/c^2$ , yielding a 4.6% larger predicted shadow radius under the minimal exponential completion [8]. At the photon sphere,  $\psi(r_{\text{ph}}) = 1$  exactly; extrapolation of the exterior solution to this point strictly exits the formal  $\psi \ll 1$  domain, and a fully non-linear solution may modify the numerical coefficient while preserving the sign of the deviation.

### D. Gravitational redshift bounded at photon sphere

A direct consequence of the finite lapse at the photon sphere: the maximum gravitational redshift for light emitted by static observers near the DFD photon sphere and received at infinity is set by the physical-metric lapse-squared  $\mathcal{L}_{\text{DFD}} = e^\psi$ , since matter clocks couple to the physical metric. Thus

$$1 + z_{\text{max}}^{\text{DFD}} = \sqrt{\mathcal{L}_{\text{DFD}}(r_{\text{ph}})} = e^{\psi(r_{\text{ph}})/2} = e^{1/2}, \quad (14)$$

giving

$$z_{\text{max}}^{\text{DFD}} = \sqrt{e} - 1 \approx 0.649, \quad (15)$$

whereas in GR the redshift diverges ( $z \rightarrow \infty$ ) for photons climbing out of a potential well approaching the horizon.

This is a clean strong-field discriminator: DFD predicts a finite upper bound on gravitational-redshift signatures from the innermost accretion regions of black holes, while GR predicts none. AGN iron  $K\alpha$  lines and quasar line profiles, which sample the innermost stable circular orbit region, could in principle constrain this bound.

## V. NO WORMHOLE AND NO EXOTIC MATTER

The principal prior alternative to GR that produces the exponential refractive form is the Yilmaz metric [24], interpreted by Boonserm, Ngampitipan, Simpson, and Visser [30] as a traversable wormhole with exotic matter. This section establishes that DFD's fundamental flat- $\mathbb{R}^3$  formulation does not carry the wormhole or exotic-matter interpretation.

### A. Yilmaz: wormhole throat at $r = m$

The Yilmaz exterior

$$ds_{\text{Yilmaz}}^2 = -e^{-2m/r} c^2 dt^2 + e^{+2m/r} [dr^2 + r^2 d\Omega^2] \quad (16)$$

has a curved 3-geometry on a constant- $t$  slice: the spatial metric is  $dl_{\text{Yilmaz}}^2 = e^{2m/r} [dr^2 + r^2 d\Omega^2]$ , giving the areal radius  $R_{\text{Yilmaz}}(r) = r e^{m/r}$ . Differentiating,  $dR/dr = e^{m/r} (1 - m/r)$  vanishes at  $r = m$ ; the second derivative there is positive ( $e/m$ ), so  $r = m$  is a minimum with  $R_{\text{min}} = em \approx 2.718m$ . Both  $r \rightarrow 0$  and  $r \rightarrow \infty$  give  $R \rightarrow \infty$ , so  $r = m$  is a wormhole throat connecting two asymptotic regions. Boonserm *et al.* [30] show that the Einstein tensor of (16) requires a stress-energy tensor violating the null energy condition at the throat. See also Hochberg and Visser [31] and Visser's systematic treatment [32] of wormhole geometries and their energy-condition violations.

### B. DFD: no throat in flat $\mathbb{R}^3$

DFD postulates flat Euclidean 3-space. The spatial metric is  $dl_{\text{DFD}}^2 = dr^2 + r^2 d\Omega^2$ , giving areal radius  $R_{\text{DFD}}(r) = r$  and  $dR/dr = 1$  for all  $r > 0$ . No critical point exists, so no throat exists on the fundamental spatial slice.

*a. Why analog gravity does not immediately refute this.* Analog-gravity constructions [20–22] demonstrate that emergent causal structure can exist on fundamentally flat substrates: acoustic horizons trap phonons in flowing fluids even though the laboratory spatial geometry is trivially flat. The claim above is therefore specifically about the *spatial* 3-metric of the DFD substrate, not a universal flat-background claim. What does DFD's derived optical metric  $\tilde{g}_{\mu\nu} = \text{diag}(-c^2/n^2, 1, 1, 1)$  say? Its spatial part is the same flat  $\mathbb{R}^3$ , so the derived optical

areal-radius function is unchanged from  $R_{\text{DFD}}(r) = r$ . No throat arises in either the substrate or the optical metric.

A stronger question — whether DFD could be reformulated in a conformally rescaled frame whose spatial part resembles Yilmaz’s — is a matter of frame choice and does not alter the physical dynamics on the flat substrate. We follow the analog-gravity convention of referring topological claims to the physical substrate metric.

### C. No exotic matter in the field equation

DFD’s field equation (3) is an elliptic PDE sourced by ordinary matter density  $\rho_m \geq 0$ . The energy density of the  $\psi$  field is  $u_\psi = (c^4/8\pi G)W(|\nabla\psi|^2/a_*^2)$  with  $W(s) = s - \ln(1+s)$ , which reduces to  $u_\psi = (c^4/8\pi G)|\nabla\psi|^2$  in the  $\mu \rightarrow 1$  regime. Both  $W$  and its  $\mu \rightarrow 1$  limit are nonnegative, so  $u_\psi \geq 0$  for all configurations. There is no Einstein tensor to balance against a stress-energy tensor, no Hilbert action, and no requirement for energy-condition-violating matter. Internal consistency of (3) is a standard question for monotone elliptic operators, resolved by standard PDE theory [8].

## VI. WHY GR IS A PADÉ APPROXIMANT OF DFD SPECIFICALLY

The Padé identity (9) relates GR’s squared inverse-lapse to  $\exp(2u)$ . Any theory whose lapse-squared (or equivalent clock-rate scalar) takes the form  $\exp(2u)$  will satisfy the same Padé identity with GR. This includes Yilmaz [24], Puthoff’s polarizable vacuum [5, 6], and Broekaert’s scalar construction [7]. The Padé identity is therefore not *mathematically* specific to DFD.

The statement “GR is the  $[1, 1]$  Padé approximant of theory  $X$ ” becomes physically meaningful, however, only when theory  $X$  is outside GR — that is, when the statement expresses a genuine inter-theory reduction rather than a tautology.

*Yilmaz*: the metric (16) is a solution of Einstein–Klein–Gordon equations, embedded *within* GR. The statement “GR is a Padé approximant of Yilmaz” is therefore self-referential: it says GR approximates a particular GR solution, which is trivially true for any limit procedure.

*Puthoff polarizable vacuum*: formulated outside standard GR, with a scaled-ruler/scaled-clock heuristic. The Padé identity applies and expresses a genuine reduction, but PV lacks a nonlinear field equation and a topological foundation; it is closer to an effective phenomenology than a complete theory.

*DFD*: formulated on flat Euclidean  $\mathbb{R}^3$  with a scalar loading field, a nonlinear elliptic field equation (3) of AQUAL type, and a topological foundation on  $\mathbb{C}P^2 \times S^3$  [8]. Not a GR solution.

The Padé identity, combined with the ontological asymmetry, establishes DFD as a non-trivial candidate for “the fundamental theory of which GR is an approximation.” The identity does not prove this selection — only experiment can. Among the candidate exponential theories surveyed, DFD is the one whose status as a Padé parent of GR is not tautological.

*Asymmetry statement*. If reality follows DFD’s exponential, GR’s post-Newtonian success is mathematically inevitable at measured precision: the  $[1, 1]$  Padé agrees with the exponential through  $O(u^2)$ , which covers all current clock-rate-based solar-system tests. Conversely, if reality follows GR strictly, DFD’s predictions disagree only at the strong-field scales probed by Event Horizon Telescope observations and future precision gravitational-wave measurements. The distinction is experimentally accessible; Sec. VII surveys the program.

### A. The Padé hierarchy: GR as $m = 1$ , DFD as $m \rightarrow \infty$

The  $[1, 1]$  identity (9) is the first nontrivial member of an infinite hierarchy of rational approximants. For each  $m \geq 1$ , the diagonal  $[m, m]$  Padé of  $\exp(u)$  is the unique rational function of numerator and denominator degrees both equal to  $m$  whose Taylor series matches  $\exp(u)$  through  $O(u^{2m})$ . Its square therefore satisfies

$$[P_{m,m}(\exp(u))]^2 = \exp(2u) + O(u^{2m+1}). \quad (17)$$

Each successive  $m$  matches the exponential one Padé order further out in the lapse-sector scalar. This defines a hierarchy of rational functions in  $u$  whose weak-field expansions agree with  $\exp(2u)$  to increasing order, not a hierarchy of full metric theories: the lapse-squared scalar does not *per se* imply matching of ray-optics or spatial-curvature PPN content at higher orders.

*a. The hierarchy explicitly*. Table III lists the matching order and first real pole location for  $m = 1$  through 5. The  $[m, m]$  Padé of  $\exp(u)$  has the closed form

$$N_m(u) = \sum_{k=0}^m \binom{m}{k} \frac{(2m-k)!}{(2m)!} u^k, \quad (18)$$

$$D_m(u) = N_m(-u), \quad (19)$$

with  $P_{m,m}(\exp(u)) = N_m(u)/D_m(u)$ .

*b. Interpretation: GR is the  $m = 1$  case*. The identity (9) establishes that the squared inverse-lapse of isotropic-coordinate Schwarzschild equals  $[P_{1,1}(\exp(u))]^2$ . This positions Schwarzschild’s lapse sector as the  $m = 1$  element of the hierarchy (17). Every finite- $m$  truncation has its own lapse pole at finite  $u$ ; only the entire function — the  $m \rightarrow \infty$  limit — has no pole at any finite  $u$ .

TABLE III. The Padé hierarchy of squared diagonal Padé approximants of  $\exp(u)$  in the lapse-squared scalar  $u = GM/(\rho c^2)$ . Each finite- $m$  member is a rational function of  $u$  whose Taylor expansion agrees with the target  $\exp(2u)$  through  $O(u^{2m})$ , with a Padé pole on or near the positive real axis at the indicated location.

$m$	Matches $\exp(2u)$ through	First real pole
1	$O(u^2)$	$u = 2$ (GR Schwarzschild)
2	$O(u^4)$	none (complex pair only)
3	$O(u^6)$	$u \approx 4.64$
4	$O(u^8)$	none (complex pair only)
5	$O(u^{10})$	$u \approx 7.29$
$\vdots$	$\vdots$	$\vdots$
$\infty$	exact (entire)	none (DFD)

*c. The  $m \rightarrow \infty$  limit recovers DFD.* The sequence of squared diagonal Padé approximants converges to the entire function  $\exp(2u)$  pointwise for all finite  $u$  and uniformly on any compact subset of the complex plane avoiding the poles of  $P_{m,m}$  [49]. In the limit:

$$\lim_{m \rightarrow \infty} [P_{m,m}(\exp(u))]^2 = \exp(2u) = \mathcal{L}_{\text{DFD}}(u). \quad (20)$$

Equation (20) is the precise sense in which GR is a special case of DFD: GR's lapse-squared is the simplest rational truncation of DFD's exponential, and DFD is the unique entire-function completion of that truncation. The Schwarzschild horizon at  $u = 2$  is the Padé pole of the  $m = 1$  truncation; higher- $m$  truncations push their poles outward; only the entire function  $\exp(2u)$  has no finite pole at all. GR occupies the  $m = 1$  slot of a hierarchy whose  $m \rightarrow \infty$  limit is DFD.

*d. Scope of the reduction.* This identity operates on a single scalar function of  $u$ , the lapse-squared (which, on DFD's exterior solution, coincides numerically with the optical refractive index  $n_{\text{DFD}}$ ). The hierarchy is not a parameter limit of field equations: DFD's elliptic PDE on flat  $\mathbb{R}^3$  and GR's  $G_{\mu\nu} = 8\pi GT_{\mu\nu}/c^4$  on a curved Lorentzian 4-manifold are not connected by any single-parameter limit on either side. Full PPN agreement between DFD and GR — including the spatial-curvature parameter  $\gamma = 1$  and the ray-optics sector — is established separately via DFD's physical metric  $g_{ij} = e^{+\psi} \delta_{ij}$  in [8], and is not a consequence of the present hierarchy.

*e. Experimental implication.* All gravitational-redshift and clock-rate observations to date are consistent with the lapse-sector identity and with both GR and DFD. The structural distinction is the *absence of a finite-radius horizon*: the entire function  $\exp(2u)$  has no finite pole; every rational truncation does. The black-hole shadow (Sec. VII) and the bounded gravitational redshift at the photon sphere (Sec. IVD) probe this structural feature directly. The Padé hierarchy recasts horizons as artifacts of rational truncation in the lapse function; their absence in DFD is the empirical

lever separating the full exponential from all its rational approximants.

## VII. OBSERVATIONAL CONSEQUENCES

### A. Black hole shadow: M87\* and Sgr A\*

Using the explicit DFD exterior solution, the critical impact parameter ratio is  $b_{\text{crit}}^{\text{DFD}}/b_{\text{crit}}^{\text{GR}} = 2e/(3\sqrt{3}) \approx 1.046$ , giving a 4.6% larger geometric shadow than Schwarzschild. For M87\* [35] with observed ring diameter  $42 \pm 3 \mu\text{as}$ , DFD predicts  $\sim 43.9 \mu\text{as}$  ( $0.6\sigma$  consistency). For Sgr A\* [36, 37] with observed shadow diameter  $51.8 \pm 2.3 \mu\text{as}$ , DFD predicts  $\sim 54.2 \mu\text{as}$  ( $\sim 1.1\sigma$  tension). The shadow-deviation parameter  $\delta$  defined by Kocherlakota and Rezzolla [39] gives  $\delta_{\text{DFD}} \approx +0.046$ , compatible with VLTI's constraint ( $\delta \in [-0.17, 0.01]$ ) at  $\sim 1.4\sigma$  and with the Keck constraint at  $\sim 0.5\sigma$ . Under current data, M87\* mildly favors DFD while Sgr A\* mildly favors GR; the combined tension is at most  $\sim 1\sigma$  and does not statistically discriminate. Next-generation facilities [38, 40, 41] are expected to reach the required precision.

### B. Gravitational-wave ringdown and echoes

DFD's horizonless exponential profile admits a modified near-photon-sphere potential that in principle supports reflected modes, distinct from exotic compact objects with sharp reflecting walls (gravastars, boson stars, fuzzballs) [42]. A DFD-specific prediction would be an effective reflectivity  $|\mathcal{R}|^2$  determined by the gradient of  $\psi$  near the photon sphere rather than by a postulated surface reflector. Because the exponential profile is smooth, the expected reflectivity is substantially lower than the ECO-class signatures already constrained by LVK searches [43–46]. A quantitative DFD echo spectrum requires the full nonlinear  $\psi$  profile around a compact source and is deferred to numerical studies. No echo signals have been observed to date.

### C. Gravitational-wave memory and distinguishing tests

Both DFD and GR reduce to linearized gravity in the far-field. The Christodoulou memory effect in DFD, computed from time-dependent sources of  $\psi$  on flat  $\mathbb{R}^3$ , agrees with the GR result to the precision probed by LIGO-band stellar-mass binary black-hole sources [8, 47]. Measurements sensitive to wormhole topology [48] or exotic-matter throat structure would distinguish Yilmaz-type scenarios from both DFD and GR; DFD predicts no such signatures.

### VIII. CONCLUSION

General relativity is the simplest rational truncation of Density Field Dynamics in the gravitational clock-rate sector. The isotropic-coordinate Schwarzschild squared inverse-lapse equals the squared  $[1, 1]$  Padé approximant of DFD's exponential exactly; the Schwarzschild horizon at  $r = 2GM/c^2$  is the Padé pole of that  $m = 1$  truncation. DFD is the entire-function completion of the same hierarchy, free of any finite-radius pole, with  $r = 2GM/c^2$  appearing as a photon sphere rather than a horizon. The two theories agree through  $O(u^2)$  — reproducing the post-Newtonian coefficient  $\beta = 1$  and consistent with all current gravitational-redshift observations — and first differ at  $O(u^3)$ . The PPN parameter  $\gamma$  and ray-optics observables, which depend on the spatial metric, are established separately for DFD via its physical metric [8].

The exponential refractive form is not novel in isolation: constructions due to Papapetrou, Yilmaz, Dicke,

Puthoff, and Broekaert all reach exponential forms. The Padé identity relating GR's squared inverse-lapse to any such exponential is, to our knowledge, novel as a named relation. Its structural content is that an entire function (the exponential) is being approximated by a rational function (the Padé) of order  $[1, 1]$ , with the approximation's pole appearing as the approximated form's horizon coordinate.

Among the candidate exponential theories, DFD is the one outside GR whose Padé relation to GR expresses a non-tautological inter-theory reduction: Yilmaz is a GR solution, DFD is not. This ontological asymmetry is what makes the statement “GR is a Padé approximant of DFD” physically meaningful.

Strong-field observations — black-hole shadow sizes, gravitational-wave ringdown spectra, near-horizon spectroscopic signatures of bounded gravitational redshift ( $z_{\max} \approx 0.65$  in DFD vs.  $z \rightarrow \infty$  in GR) — offer the empirical program to distinguish DFD from GR. Current data is consistent with both at the  $\sim 1\sigma$  level.

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